Discourse on Paradoxes with emphasis on Russell's Paradox.

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Abstract

This paper attempts to further the understanding of paradoxes by explaining their past and present. The history of the major paradoxes is traced and special emphasis is made of Russell's paradox and Set theory. Consequences of the paradox in question with relation to general mathematical science are also mentioned.

1 Introduction

In this paper, we propose to discuss initially what a paradox is! We shall go through the history of a paradox, how the earliest minds evolved a paradox and its effects in the age when mathematics itself was not as evolved. We shall look at paradoxes evolved which affected various branches of science and maths e.g. Zeno's paradoxes. To understand the main topic of this paper we shall discuss the basics of set theory, and then we shall tackle the issue of the paradox itself and its variants. Because of the importance of this paradox, the consequences will be debated to a large degree as well. In the end, we shall attempt to put *Russell's Paradox* in perspective.

2 Background

2.1 A history of Paradoxes

> **Homer Simpson**: Can God microwave a burrito so hot, that God could not eat it?
> **Ned Flanders**: ummm, errr that's a ho diddly ho of a pickle Homer!
> **Homer Simpson**: Now you know what I have been going through.

Can God create a stone too heavy for God to lift? What came first, the chicken or the egg? Questions like these have been the staple of philosophers and art students for generations. These are called paradoxes.

A paradox is a statement which appears to be either true or false, but an extension of the original statement, ends up nullifying the previous state of truth or falsification. For example, God, being all powerful, obviously can create a stone of any size, mass or weight and hence could create a stone large enough even for God, but if God then, could not lift the stone, then the statement about God being all powerful is false.

Figure 1: Chicken or Egg?

Not all paradoxes are fun and games, most involve serious mathematics and have tangible mathematical solutions and consequences. One of the earliest paradoxes discovered is called the *Sorites Paradox* or *The Paradox of the Heap.* Although, it has been expressed in many forms and variations, the most original is as follows:

> Suppose there is a huge mound of sand. And suppose that this huge mound, is a heap. If one removes one grain of sand from the heap, then one is still left with a heap of sand. If we continue to remove grains of sand from the heap, then via the first principle, we can conclude that even a grain of sand constitutes a heap.

The above paradox is called a *Paradoxical Argument* and is part of a set of paradoxes or arguments called the *Little-by-little Arguments.* These arose from the indeterminacy surrounding limits of applications of the predicates that these involved. The phenomenon at the heart of this paradox and for similar ones, is called the phenomenon of *Vagueness.*

Sorites arguments of the paradoxical form are to be distinguished from multi-premise syllogisms (poly-syllogisms) which are sometimes also referred to as Sorites arguments. Sorites paradox and other similar ones are often attributed to the Megarian logician **Eubulides of Miletus**.

Another set of famous paradoxes are called the *Zeno's paradoxes.* These are attributed to the philosopher *Zeno of Elea.* Zeno devised these problems to support the claim of another philosopher *Parmenides's* doctrine that **All is one**, contrary to what our senses tell us. Even though Zeno composed eight paradoxes, three have become more famous than the others and are called the *Paradoxes of Motion.*

- ***The Achilles Paradox* - One can never catch up**

  In a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead

- ***The Dichotomy Paradox* - One cannot even start**

  That which is in locomotion must arrive at the half-way stage before it arrives at the goal.

- ***The Arrow paradox* - One cannot even move**

> If everything when it occupies an equal space is at rest, and if that which is in locomotion is always occupying such a space at any moment, the flying arrow is therefore motionless.

What Zeno tried to show was that it was really hard to compute something that is infinite or something whose solution depends upon a series of infinite steps. This problem appeared to be solved when Sir Isaac Newton developed Calculus. And through it, there were a series of solutions proposed for Zeno's paradoxes using the concept of *Limit*. Even though this leads to an eventual solution of the problem, there lies an inherent flaw with using calculus for this paradox.

Using Limits, calculus gives us an answer to the paradox, but it does not answer how the series is to be completed and the limit ever reached. The problem can be defined as that calculus finds the solution but never explains how the infinite number of steps involved in reaching the solution were ever done, which according to Zeno, is the actual paradox.

There have been other solutions to these paradoxes as well such as assuming that time and space are atomic and hence cannot be divided in the way Zeno attempts to do in his paradoxes. However these paradoxes have been assumed to be solved using Infinite Sequence solutions developed by *Newton* and *Leibniz* and Calculus which was developed by the former.

Even though these paradoxes might seem trivial, especially Zeno's, it has had a few applications in modern science, notably the *Quantum Zeno Effect*.

The *Quantum Zeno Effect* states that

> Dynamical Evolution of a quantum system can be hindered (or even inhibited) though observation of the system and is strongly reminiscent of the Arrow Paradox.

One of the most ancient legal problems arises from Ancient Greece and is known as the *Paradox of the Court*. A teacher *Protagoras* took a pupil *Euathlus* and they had an agreement that the latter would pay the former after the latter had won his first case.

Euathlus never got clients and hence was never obliged to pay his master, who sued him for not paying him. Protagoras argued that should he win the case, then the latter would have to pay him but if he lost, then Euathlus would have won his first case and would still have to pay him. Euathlus argued that should he win, the court would allow him not to pay, but if he lost, then he still would not have won a case and hence would not have to pay Protagoras anything.

However, a prominent paradox arising from Set theory was the *Burali-Forti Paradox*. But before we begin talking about the paradoxes that arise in Set theory, we must elaborate a bit about the theory itself.

2.2 Set Theory

Set theory is a mathematical theory which represents collections of objects. These objects can be concrete objects such as things that you see and touch, for example cars, or it can be a collection of things that are abstract, for example all shapes. Set theory branched into two forms in mathematics. One is called the *Naive Set Theory* and the other is called the *Axiomatic Set Theory*.

The Naive Set theory simply discusses about Sets in informal language and a few of their properties are investigated like Universal nature of sets, cardinality, membership, equality etc. This theory also proposed any operation on any set without any kind of restriction which led to a series of paradoxes.

Because of the simplistic nature of this theory, a few paradoxes arose in Set theory itself, and hence a more rigorous definition had to be developed and it was named Axiomatic Set Theory. This was a more strict definition of Set Theory in first order logic.

Some basic definitions of Set Theory are :

- **Membership**
  Let there be a set X. And let X contain an element A. Then A is a member of X and it is denoted by $A \in X$.

- **Subset**
  Let A be a set and let B be a set. If every element of A exists in B, then A is a subset of B. If A is a subset of B and B has more elements than A, then A is a proper subset of B and is denoted by $A \subseteq B$. If A and B contain the same elements and the same number of elements, then A is an improper subset of B and is denoted by $A \subsetneq B$. B is known as the super set of A.

- **Ordinal Sets**
  A set A is an ordinal set if and only if A is totally ordered with respect to self containment (subset relationship) and that if $x \in A$ then $x \subseteq A$.

- **Union**
  Let there be a set A and a set B. Then the union operator, denoted by $\cup$ is defined as $A \cup B := \{x : \{x \in A\}\ or\ \{x \in B\}\}$

- **Intersection**
  Let there be a set A and a set B. Then the intersection operation, denoted by $\cap$ is defined as $A \cap B := \{x : \{x \in A\}\ and\ \{x \in B\}\}$

- **Difference**
  Let there be a set A and a set B. Then the difference between A and B is defined as A $B := \{x : \{x \in A\}\ not\ \{x \in B\}\}$

There are also a few important predefined sets such as sets if Natural Numbers $\mathbb{N}$, set of Integers $\mathbb{Z}$ etc.

2.3 Predecessors of Russell

The Burali-Forti paradox was discovered by *Cesare Burali-Forti* in 1897. The paradox demonstrated that generation of a set of all ordinal numbers using the naive set theory creates a contradiction and shows a flaw in the process of its generation. This can be denoted as the following:

Let ϕ be a set of all ordinals. But ϕ will also contain all the properties of ordinal numbers and this leads us to create a successor to ϕ which is $\phi + 1$. Since this is a successor of ϕ, it is an ordinal number itself and hence must be a member of ϕ and hence we arrive at $\phi < \phi + 1 \leqslant \phi$.

Another Basic theorem worth mentioning is the *Cantor's Theorem* which states that the power set (set of all subsets) of any set has a greater cardinality (number of elements in the set) than the set itself. For example, if X is the power set of Y, then X has a strictly greater number of elements than Y. Cantor's theorem, surprisingly not only holds for finite sets but for infinite sets as well.

It was from all these paradoxes, from the evolution of the naive set theory, that led to the development of Russell's paradox

3 Russell's paradox

It was in the year 1901 that a gentleman named *Bertrand Russell* was working on Cantor's theorem and its subsequent analysis when he came across a paradox. In layman's terms, the paradox can be stated as

> Let X be a set of all leprechauns. Since X itself is not a leprechaun, X will not contain itself as a member. Hence we can call X a *Normal Set.* Now suppose Y is a set of all things that are not leprechauns. Since Y is not a leprechaun, hence it does contain itself as a member, and is called an *Abnormal Set.*
>
> Now consider a set G which has its elements, all normal sets. Now, if G is normal, then it should contain itself as a member, but the moment it contains itself, it becomes abnormal. And therein lies the paradox.

Russell probably discovered it around 1901 and it was first published in the *International Monthly* in 1901 under the heading **Recent Work in Philosophy of Mathematics**. Russell wrote to Frege about his Paradox just as the latter was to publish his work on Set Theory and mathematical philosophy or *Grundgesetze der Arithmetik.* Frege was most worried that this paradox would not sit well with his other suppositions and therefore he hurriedly wrote his own solution to the paradox while admitting to its presence. However, that hurriedly composed solution was not sufficient to solve the paradox.

Many noted mathematicians of the day commented and wrote upon this paradox. For example *Ludwig Wittgenstein* tried to dispose of this paradox by giving his counter proof in his book **Tractatus Logico-Philosophicus**, while *Ernst Zermelo* of the ZF-Set theory fame also noted that the paradox existed, but would not comment on it any further.

The significance of Russell's paradox is that it has been developed from the idea that any logically coherent condition can be used to obtain a set. The problem lies within naive set theory itself and its unique component called the *Comprehension* or the *Abstraction Axiom.* This axiom states that any function F with an independent variable x can determine a set. Because of the loose nature of the rules governing the creation and manipulation of sets in the naive set theory, this and many paradoxes arose. Consequently many solutions were proposed and we shall look at them later.

Even as Russell's paradox was being debated and analysed, there were many variants of the paradox in vogue. Here are a few of them

- *The barber's paradox*

  This paradox was related by Bertrand Russell himself and it can be stated as

  > There was once a barber and he lived in a town. He only ever shaved those men who could not shave themselves and in that town, men either shaved themselves or were shaved by the barber. So who shaved the barber?

  The logical train of thought can be pursued by presuming that the barber did shave himself. But then the statement tells us that he only shaved those who could not shave themselves. This leads us to believe that the barber did not shave himself. But the same statement then again informs us that the people in that town either shave themselves or are shaved by the barber . Hence this paradox is a closely related version of the set theory version of Russell's paradox.

  Over the years, many people have tried and create loopholes out of this particular case by stating that the barber could be a woman or that the barber could be from out of town, but none of these suggested loop holes have been granted official recognition.

- *The Liar Paradox*

Imagine someone saying this to you : **I am lying now**. Because the person says that he is lying, one cannot assume that the said person is lying now or not and is hence a paradox in itself.

- *The Grelling Nelson Paradox*

  The Grelling Nelson paradox was discovered by Messrs. Kurt Grelling and Leonard Nelson in the year 1908. It is noted for its similarity to Russell's Paradox.

  The paradox can be stated as :

  > Let an adjective be autological if and only if it refers to itself. And an adjective is heterological if and only if it does not describe itself. Now knowing this, we can ask, is the word *heterological* a heterological word? If yes, then it is autological, which is a contradiction. But even if it isn't , then heterological is heterological word which again is a contradiction.

  Even though this paradox can be eliminated by slightly changing the definition of heterological, the renewed definition is still subject to the same paradox.

Other examples include *Richard's Paradox* which is now an important distinctive tool between mathematics and metamathematics.

4 Consequences of Russell's paradox

One of the most obvious conclusions that was reached after the publication of Russell's Paradox was that every logician decided that the definition and thereby the interpretation of a *set* could not be allowed to be naive. It was not a coincidence that most paradoxes involving set theory had emerged from the naive branch of that math. This was so because of the leniency of rules governing that branch. And this was noted by Ernst Zermelo when he stated the Zermelo set theory.

4.1 Zermelo's Theory

Zermelo wanted to root out the said antimonies or paradoxes from set theory by making it more defined and constructing stricter rules or axioms to govern it. This came to be known as the *Axiomatic Set Theory.* The aim of Zermelo's theory was to provide axioms that would reduce Cantor's definition to a concise set of few axioms, even if they may not be consistent.

The Axioms of Zermelo's Theory are :

- **Axiom of extensionality**
  Every set shall be defined by it's elements

- **Axiom of Elementary Sets**
  There exists an empty set ϕ which has no elements.
  If A is an object in a domain, then it contains a set {A} etc.

- **Axiom of Separation**
  A set 'X' contains a subset 'X_' which contains elements of a propositional function which is definite for all elements of X.

- **Axiom of a Power set**
  For every set X, there will be set that will contain as its elements, all the subsets of X.
- Axiom of Union
  For every set X, there will be set which contains each end every element of the elements of X.
- Axiom of Infinity
  If X is a set with only one member, ϕ as its element, then if later on 'y' is an element if X, then the union set of X's elements will also be a member of X.

Of all of these Axioms, it was the *Axiom of Separation* which the author of this paper and theory thought would eliminate the paradoxes. Even though Cantor had also given a similar axiom, this was different in the sense that because it refused to allow sets to be defined in any arbitrary way. Because this axiom forced the separation of sets as subsets of already given sets, this , according to Zermelo, eliminated the contradictions that arose from set of all sets and all paradoxes associated with that phenomena. However, concise as this theory was, it was still incomplete, because it did not address the issue of ordinality. Also, in Zermelo's axioms, Zermelo mentioned the property "definite" with regards to a set but it was not clearly explained in itself or with reference to context.

4.2 Zermelo-Fraenkel Theory

Because of the shortcomings of the Zermelo Theory, there emerged another theory which bound set theory even tighter with stricter rules. This theory, called ZFC, is the standard form of axiomatic set theory and hence has become one of the founding bases of modern set theory.

Even though in this paper, both ZT and ZFT have been shown to contain a certain number of axioms, but in reality, they can contain infinite number of axioms and his has been proved by Richard Montague (1961) and to counter this, the *Von Neumann-Bernays-Gdel set theory* was developed by John Von Neumann in the 1920's, modified by Paul Bernays in 1937 and simplified by Kurt Godel in 1940. This set theory has a finite number of axioms as compared to the ZFC.

The Axioms most prevalent in this theory are :

- **Axiom of Extensionality**
- **Axiom of Foundation**
- **Axiom of Specification**
- **Axiom of Pairing**
- **Axiom of Union**
- **Axiom of Choice**

Even though these ones are the most noted ones, one must always remember that Zermelo-Fraenkel Set theory, along with the Zermelo set theory can have an infinite number of axioms.

Even though the ZF set theory made set definition and construction stricter that what they were in naive set theory, it still had some flaws.

- It was too strong. In fact, it was more strong and strict than was actually required in day to day mathematics

- It does not admit to the existence of a universal set, because doing so would again result in *Russell's paradox*

Because of the above rules and axioms, ZFC does not fall prey to the three major paradoxes, Russell. Cantor and the Burali Forti.

4.3 Von Neumann-Bernays-Gödel Set Theory

Another Set theory that emerged in the aftermath of the paradoxes was developed by Von Neumann, modified by Bernays and simplified by Gödel. Unlike in ZFC, this set theory, called NBG, is the more conservative version of the former. As proved by Richard Montague in 1961 (mentioned previously), NBG can have a finite number of axioms and also unlike ZFC, it does mention the case for a universal set or a class.

This theory has the same basic idea as that of naive set theory and on an initial glance it seems that it would fall prey to the same contradictions that naive set theory suffered from. For example, the idea of classes is as follows:

There is a membership relation: $a \in S$ can be defined as S is a class and a is a set inside that class. A development of a set can be defined using Predicate Calculus in the following fashion: $\mathbf{Aa}(A, a) := \forall x(x \in A \Leftrightarrow x \in a)$. i.e. a is set of class A as long as every element in the class belongs in the set a. Because in this theory, there is a clear distinction between a set and a class, this theory does not run the risk of generating paradoxes like the naive set theory.

The axioms belonging to this theory are:

- **Axiom of Entensionality**
  If $\forall x(x \in A)$ and $\forall y(y \in B)$ and x = y, then A=B.

  Axiom of Class Comprehension
  For any formula ϕ, with no quantifiers over classes, we have $\forall x(x \in A) \Leftrightarrow \phi(x)$
- **Axiom of pairing**
  If A and B are two sets, then there is a set which has A and B as it's elements.
- **Axiom of Limitation of Size**
  A set can exist in a class only if the class and the complementary class do not share a one to one relationship with its elements.

There exist other axioms as well but they are similar to the ones expressed in ZF and ZFC

The advantage of the NBG was that even though it had finite number of axioms, and the fact that it has a defined concept of a universal set in shape of a *class of all sets*, it still manages to deviate away from the paradoxes that plagued naive set theory because of its simple axioms.

4.4 Morse-Kelley Set Theory

Another theory that evolved and bettered the understanding and evolution of sets is the Morse-Kelley Set theory. It is considered to be a first-order derivative of the Zermelo Set Theory.

In this theory, we have classes as the main objects (just like in Neumann-Bernays-Gödel) and sets are members of these primary objects.

The Axioms involved in this theory are

- **Axiom of Extensionality**
- **Axiom of Class comprehension**

- **Axiom of Pairs**
- **Axiom of Limitation of Size**
- **Axiom of Power Set**
- **Axiom of Union**
- **Axiom of infinity**
- **Axiom of Foundation**

Unlike ZF or ZFC, this set theory cannot be bound by a finite set of axioms, but the few axioms mentioned make it a bit more strict than either of the two mentioned.

In NBG, we saw in the *Axiom of class comprehension* that the formula ϕ extended over classes with no quantifiers whereas in this theory, and hence that does not allow separation with respect to sets. Because this theory extends the aforementioned axiom less conservatively than NBG extended the one from ZFC, it makes this theory stronger.

In the previous theories, an empty set had to be defined with a special clause explaining it, but that is not required in this theory. This is taken care of by the *Axiom of Limitation of Size* and the fact that while acknowledging the existence of a universal set, this theory also states that the universal set shall never be empty.

4.5 Theory of Types

In Logic, a theory of Types is a formal system developed by logicians to serve as an alternate to the naive set theory after the latter began to be plagued by contradictions.

The first type theory to be developed was made by Bertrand Russell himself to counter his own paradox. He developed it after the solution put forth by *Frege* was shown to be inconsistent. This theory avoids the *Russell's Paradox* by creating a hierarchy of types, and then assigning each entity (mathematical and otherwise) a type and building objects from objects of a preceding type. However, Bertrand was not the only one to put forward a type theory. Mendelson was among others who put forward an axiomatic type theory.

In *Mendelson's* type theory, the axioms given were:

- **Identity**
- **Extensionality**
- **Comprehension**
- **Infinity**

The most remarkable outcome of this type theory formal system was that it led to the development of better languages and better systems and hence eventually resulted in the formation of type checking phase in compilers.

But a landmark piece of literature that developed due to all of this is called *Principia Mathematica*.

4.6 Principia Mathematica

Written by *Alfred North Whitehead* and *Bertrand Russell*, this book, was a landmark defence of logicism and was published in three volumes, in 1910, 1912 and 1913. It defended the view that all mathematics, at some level could be reduced to logic. It also managed to give detailed explanations and definitions to various views such as Set Theory and finite mathematics etc. there by making them popular beyond what the authors had hoped.

At first , it would seem that the primary view of this tome would appear difficult to say the least because it tried to propound views as logical that had uptil that time been seen as purely empirical. Also , for a book that purported to defend logicism, it introduced two axioms that were decidedly non-logical, the *Axiom of Reducibility* which basically was Russell's way of countering the weak argument of Theory of types against his own paradox and others of that ilk; and the *Axiom of Infinity* which expressed that there were infinite number of objects.

The three major outcomes of this publication where

- It popularized modern logic immensely
- It opened the idea of metalogic for researchers like Gödel by showing the power of expression of modern predicative logic.
- It reaffirmed connections between logic, metaphysics and epistemology and induced fresh research and analysis into all three areas.

5 Discussion & Conclusion

Uptil now, we have seen that most of the paradoxes that emerged have emerged from Set theory in its naive form. In the begining of the logicism, a set could be defined simply as an arbitrary collection of objects, there was no major concept of class or a universal set.

When the concept of a set of sets came about, so came the paradoxes and the various definitions of it. However, we see that the evolution of paradoxes was necessary. It might have slowed progress at the time of discovery, an example being the delay of publication of Frege's book due to Russell's paradox, but over all, these have enhanced the field of logic and theory.

As more and more paradoxes came into being , more and more theories and many different kinds of theories came into being each contributing to the development and enhancement of logic. However, for the development of Set Theory and its evolution into Modern Set theory and its present form, Russell's Paradox has to be attributed tremendously.

The beauty of this paradox is not that it comes up with a clever contradiction. The point is that this paradox challenges the very basic definition of a set and it's properties. Once this paradox was discovered, *Naive Set Theory* was no longer feasible. Stricter rules had to be developed and formal axioms had to be stated and this led to the development of the previously mentioned formal systems for example, Zermelo, Zermelo-Fraenkel etc.

Because of the way it challenged the set theory prevalent at that time, and because of the systems and axioms that had to be developed to solve this paradox and because of the subsequent development that resulted because of it, Russell's paradox has to be one of the most important statements in the history of logic.

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